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Research Article

Subsequential Convergence Conditions

İbrahim Çanak, Ümit Totur, and Mehmet Dik

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Let (u_n) be a sequence of real numbers and let L be any $(C,1)$ regular limitable method. We prove that, under some assumptions, if a sequence (u_n) or its generator sequence $(V_n^{(0)}(\Delta u))$ generated regularly by a sequence in a class $\mathcal{A}$ of sequences is a subsequential convergence condition for L, then for any integer $m \geq 1$, the mth repeated arithmetic means of $(V_n^{(0)}(\Delta u))$, $(V_n^{(m)}(\Delta u))$, generated regularly by a sequence in the class $\mathcal{A}^{(m)}$, is also a subsequential convergence condition for L.

1. Introduction

Let (u_n) be a sequence of real numbers. Let c_o, ℓ_∞, $\mathcal{S}$, and $\mathcal{M}$ denote the space of sequences converging to 0, bounded, slowly oscillating, and moderately oscillating, respectively.

The classical control modulo of the oscillatory behavior of (u_n) is denoted by $\omega_n^{(0)}(u) = n\Delta u_n$, where $\Delta u_n = u_n - u_{n-1}$ and $u_{-1} = 0$ and the general control modulo of the oscillatory behavior of integer order m of (u_n) is defined [1] inductively by $\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n^{(1)}(\omega^{(m-1)}(u))$, where $\sigma_n^{(1)}(u) = (1/(n+1))\sum_{k=0}^{n} u_k$.

The Kronecker identity $u_n - \sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u)$, where $V_n^{(0)}(\Delta u) = (1/(n+1))\sum_{k=0}^{n} k\Delta u_k$, is well known and used in various steps of proofs of theorems. For each integer $m \geq 1$ and for all nonnegative integers n, we inductively define sequences related to (u_n) such as $V_n^{(m)}(\Delta u) = \sigma_n^{(1)}(V^{(m-1)}(\Delta u))$ and $\sigma_n^{(m)}(u) = \sigma_n^{(1)}(\sigma^{(m-1)}(u))$, where $\sigma_n^{(0)}(u) = u_n$.

Throughout this work, a different definition of slow oscillation better tailored for our purposes will be used. A sequence $u = (u_n)$ is slowly oscillating [2] if $\lim_{\lambda \to 1^+} \overline{\lim}_n \max_{n+1 \leq k \leq [\lambda n]} |u_k - u_n| = 0$, where $[\lambda n]$ denotes the integer part of λn. See [3, 4] for more on slow oscillation. A sequence $u = (u_n) \in \mathcal{S}$ if and only if $(V_n^{(0)}(\Delta u)) \in \mathcal{S}$

and $(V_n^{(0)}(\Delta u)) \in \ell_\infty$ (see [5]). A sequence $u = (u_n)$ is moderately oscillating [2] if for $\lambda > 1$, $\overline{\lim}_n \max_{n+1\leq k\leq[\lambda n]} |u_k - u_n| < \infty$. It is proved in [5] that if a sequence $u = (u_n) \in \mathcal{M}$, then $(V_n^{(0)}(\Delta u)) \in \ell_\infty$.

A sequence $u = (u_n)$ is Abel limitable to s if the limit $\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} u_n x^n = s$ and $(C,1)$ limitable to s if $\lim_n \sigma_n^{(1)}(u) = s$.

Let L be any limitation method. If $u = (u_n)$ is L limitable to s, we write $L - \lim_n u_n = s$. The limitation method L is said to be regular if $\lim_n u_n = s$ implies $L - \lim_n u_n = s$. The limitation method L is said to be $(C,1)$ regular if $L - \lim_n u_n = s$ implies $L - \lim_n \sigma_n^{(1)}(u) = s$. A sequence $u = (u_n)$ is called subsequentially convergent [6] if there exists a finite interval $I(u)$ such that all accumulation points of $u = (u_n)$ are in $I(u)$ and every point of $I(u)$ is an accumulation point of $u = (u_n)$.

Let $\mathcal{L}$ be any linear space of sequences and let $\mathcal{A}$ be a subclass of $\mathcal{L}$. For each integer $m \geq 1$, define the class $\mathcal{A}^{(m)} = \{(a_n^{(m)}) \mid a_n^{(m)} = \sum_{k=1}^{n}(a_k^{(m-1)}/k)\}$, where $(a_n^{(0)}) := (a_n) \in \mathcal{A}$. Let $u = (u_n) \in \mathcal{L}$. If

$$u_n = a_n^{(m)} + \sum_{k=1}^{n} \frac{a_k^{(m)}}{k} \tag{1.1}$$

for some $a^{(m)} = (a_n^{(m)}) \in \mathcal{A}^{(m)}$, we say that the sequence (u_n) is regularly generated by the sequence $(a_n^{(m)})$ and $(a_n^{(m)})$ is called a generator of (u_n). The class of all sequences regularly generated by sequences in $\mathcal{A}^{(m)}$ is denoted by $U(\mathcal{A}^{(m)})$. We note that $\mathcal{A}^{(0)} = \mathcal{A}$.

Tauber [7] proved that an Abel limitable sequence $u = (u_n)$ is convergent if

$$(\omega_n^{(0)}(u)) \in c_o. \tag{1.2}$$

A condition such as (1.2) is called a Tauberian condition, after A. Tauber.

Tauber [7] further proved that the condition

$$(\sigma_n^{(1)}(\omega^{(0)}(u))) \in c_o \tag{1.3}$$

is also a Tauberian condition. It was later shown by Littlewood [8] that the condition (1.2) could be replaced by

$$(\omega_n^{(0)}(u)) \in \ell_\infty. \tag{1.4}$$

Rényi [9] observed that the condition

$$(\sigma_n^{(1)}(\omega^{(0)}(u))) \in \ell_\infty \tag{1.5}$$

is no longer a Tauberian condition for Abel limitable method.

Stanojević [1] investigated behaviors of some subsequences of an Abel limitable sequence $u = (u_n)$ adding a mild condition on (u_n), together with (1.5).

Dik [6] obtained the following theorem.

Theorem 1.1. *Let (u_n) be Abel limitable and $\Delta V_n^{(0)}(\Delta u) = o(1)$. If*

$$(V_n^{(0)}(\Delta u)) \in U(\mathcal{M}), \tag{1.6}$$

then (u_n) is subsequentially convergent.

Later several improvements of Dik's theorem were obtained.

A condition that subsequential convergence of (u_n) is recovered out of its Abel limitability is called a subsequential convergence condition.

We list the subsequential convergence conditions for Abel limitable method that (1.6) can be replaced by

(i) $(V_n^{(m)}(\Delta u)) \in U(\mathcal{M}^{(m)})$ (see [10]),
(ii) $(V_n^{(0)}(\Delta u)) \in U(\ell_\infty)$ (see [6]),
(iii) $(V_n^{(m)}(\Delta u)) \in U(\ell_\infty^{(m)})$ (see [10]),
(iv) $(u_n) \in U(\mathcal{M})$ (see [11]),
(v) $(u_n) \in U(\ell_\infty)$ (see [6]).

In this work, we prove that under the assumptions if a sequence (u_n) or its generator sequence $(V_n^{(0)}(\Delta u))$ generated regularly by a sequence in a class $\mathcal{A}$ of sequences is a subsequential convergence condition for a $(C,1)$ regular limitable method L, then for any integer $m \geq 1$, the mth repeated arithmetic means of $(V_n^{(0)}(\Delta u))$, $(V_n^{(m)}(\Delta u))$, generated regularly by a sequence in the class $\mathcal{A}^{(m)}$ is also a subsequential convergence condition for L.

2. Results

Throughout this section, we require L to be $(C,1)$ regular.

We prove the following theorems.

Theorem 2.1. *For a sequence $u = (u_n)$, let $L - \lim_n u_n = s$ and $\Delta V_n^{(0)}(\Delta u) = o(1)$. If $(V_n^{(0)}(\Delta u)) \in U(\mathcal{M})$ is a subsequential convergence condition for L, then $(V_n^{(m)}(\Delta u)) \in U(\mathcal{M}^{(m)})$ for each integer $m \geq 1$ is also a subsequential convergence condition for L.*

Theorem 2.2. *For a sequence $u = (u_n)$, let $L - \lim_n u_n = s$ and $\Delta V_n^{(0)}(\Delta u) = o(1)$. If $(V_n^{(0)}(\Delta u)) \in U(\ell_\infty)$ is a subsequential convergence condition for L, then $(V_n^{(m)}(\Delta u)) \in U(\ell_\infty^{(m)})$ for each integer $m \geq 1$ is also a subsequential convergence condition for L.*

Theorem 2.3. *For a sequence $u = (u_n)$, let $L - \lim_n u_n = s$ and $\Delta V_n^{(0)}(\Delta u) = o(1)$. If $(u_n) \in U(\mathcal{M})$ is a subsequential convergence condition for L, then $(V_n^{(m)}(\Delta u)) \in U(\mathcal{M}^{(m)})$ for each integer $m \geq 1$ is also a subsequential convergence condition for L.*

Theorem 2.4. *For a sequence $u = (u_n)$, let $L - \lim_n u_n = s$ and $\Delta V_n^{(0)}(\Delta u) = o(1)$. If $(u_n) \in U(\ell_\infty)$ is a subsequential convergence condition for L, then $(V_n^{(m)}(\Delta u)) \in U(\ell_\infty^{(m)})$ for each integer $m \geq 1$ is also a subsequential convergence condition for L.*

To prove these theorems, we need the following lemma and the observation.

Lemma 2.5 [12]. *Let $u = (u_n) \in \mathcal{L}$ and $k, m \geq 0$ be any integers. If $(V_n^{(k)}(\Delta u)) \in U(\mathcal{A}^{(m)})$, then $(n\Delta)_{m+1} V_n^{(k+1)}(\Delta u) = a_n$, where $(a_n) \in \mathcal{A}$.*

Proof. If $(V_n^{(k)}(\Delta u)) \in U(\mathscr{A}^{(m)})$, it then follows that

$$V_n^{(k)}(\Delta u) = \sigma_n^{(k-1)}(u) - \sigma_n^{(k)}(u) = b_n^{(m)} + \sum_{j=1}^{n} \frac{b_j^{(m)}}{j} \tag{2.1}$$

for some $(b_n^{(m)}) \in \mathscr{A}^{(m)}$. From (2.1), we obtain

$$V_n^{(k-1)}(\Delta u) - V_n^{(k)}(\Delta u) = n\Delta b_n^{(m)} + b_n^{(m)}. \tag{2.2}$$

Subtracting (2.2) from the arithmetic mean of (2.2), we have

$$\left(V_n^{(k-1)}(\Delta u) - V_n^{(k)}(\Delta u)\right) - \left(V_n^{(k)}(\Delta u) - V_n^{(k+1)}(\Delta u)\right) = b_n^{(m-1)}. \tag{2.3}$$

Equation (2.3) can be expressed as

$$n\Delta V_n^{(k)}(\Delta u) - n\Delta V_n^{(k+1)}(\Delta u) = b_n^{(m-1)}, \tag{2.4}$$

which implies $(n\Delta)_2 V_n^{(k+1)}(\Delta u) = b_n^{(m-1)}$. By repeating the same reasoning, we have $\sigma_n^{(1)}(\omega^{(k+1)}(u)) = (n\Delta)_{m+1} V_n^{(k+1)}(\Delta u) = b_n^{(0)} = b_n$. □

For a sequence (u_n) and for each integer $m \geq 1$, we define

$$(n\Delta)_m u_n = n\Delta\left((n\Delta)_{m-1} u_n\right), \tag{2.5}$$

where $(n\Delta)_0 u_n = u_n$ and $(n\Delta)_1 u_n = n\Delta u_n$.

Observation 1 [13]. For each integer $m \geq 1$,

$$\omega_n^{(m)}(u) = (n\Delta)_m V_n^{(m-1)}(\Delta u). \tag{2.6}$$

The proof of Observation 1 easily follows from the mathematical induction.

Proof of Theorem 2.1. Assume that $(V_n^{(0)}(\Delta u)) \in U(\mathcal{M})$ is a subsequential convergence condition for L. Since $(V_n^{(0)}(\Delta u)) \in U(\mathcal{M})$, $V_n^{(0)}(\Delta u) = b_n + \sum_{k=1}^{n}(b_k/k)$ for some $(b_n) \in \mathcal{M}$. Hence, we have

$$n\Delta V_n^{(0)}(\Delta u) = n\Delta b_n + b_n. \tag{2.7}$$

Taking the $(C,1)$ mean of both sides of (2.7), we obtain $n\Delta V_n^{(1)}(\Delta u) = V_n^{(0)}(\Delta b) + \sigma_n^{(1)}(b) = b_n$. Since $(b_n) \in \mathcal{M}$,

$$V_n^{(0)}(\Delta b) = O(1) \tag{2.8}$$

by a result in [5]. Notice that (2.8) can be rewritten as $V_n^{(0)}(\Delta b) = (n\Delta)_2 V_n^{(2)}(\Delta u) = O(1)$ in terms of the sequence $u = (u_n)$. Let $(V_n^{(m)}(\Delta u)) \in U(\mathcal{M}^{(m)})$. By Lemma 2.5, $(\sigma_n^{(1)}(\omega^{(m+1)}(u))) \in \mathcal{M}$. From the last statement, we conclude that $\sigma_n^{(1)}(\omega^{(m+2)}(u)) = (n\Delta)_{m+2} V_n^{(m+2)}(\Delta u) = O(1)$, or equivalently

$$\sigma_n^{(1)}(\omega^{(m+2)}(u)) = (n\Delta)_2 V_n^{(2)}(\Delta\sigma^{(1)}(\omega^{(m-1)}(u))) = O(1). \tag{2.9}$$

It easily follows from the existence of L-limitability of (u_n) to s that

$$L-\lim_n \sigma_n^{(1)}(\omega^{(m-1)}(u)) = 0. \tag{2.10}$$

The condition $\Delta V_n^{(0)}(\Delta u) = o(1)$ implies that

$$\Delta((n\Delta)_m V_n^{(m)}(\Delta u)) = \Delta V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(m-1)}(u))) = o(1). \tag{2.11}$$

Taking into account (2.9), (2.10), and (2.11), we obtain that $(\sigma_n^{(1)}(\omega^{(m-1)}(u)))$ is subsequentially convergent. By the fact that every subsequentially convergent sequence is bounded, $\sigma_n^{(1)}(\omega^{(m-1)}(u)) = O(1)$, or equivalently

$$\sigma_n^{(1)}(\omega^{(m-1)}(u)) = (n\Delta)_2 V_n^{(2)}(\Delta\sigma^{(1)}(\omega^{(m-4)}(u))) = O(1). \tag{2.12}$$

As in obtaining (2.10) and (2.11), we also have

$$\begin{gathered} L-\lim_n \sigma_n^{(1)}(\omega^{(m-4)}(u)) = 0, \\ \Delta((n\Delta)_{m-3} V_n^{(m-3)}(\Delta u)) = \Delta V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(m-4)}(u))) = o(1), \end{gathered} \tag{2.13}$$

respectively.

Again taking into account (2.12) and (2.13), we obtain that $(\sigma_n^{(1)}(\omega^{(m-4)}(u)))$ is subsequentially convergent. Continuing in this manner, if $m \equiv 0 \pmod 3$, we have that $((n\Delta)_2 V_n^{(2)}(\Delta u)) = (\sigma_n^{(1)}(\omega^{(2)}(u)))$ is subsequentially convergent and then

$$(n\Delta)_2 V_n^{(2)}(\Delta u) = O(1). \tag{2.14}$$

Since $L-\lim_n u_n = s$, we have

$$L-\lim_n \sigma_n^{(1)}(\omega^{(2)}(u)) = 0. \tag{2.15}$$

Again it follows from the conditions $\Delta V_n^{(0)}(\Delta u) = o(1)$, (2.14), and (2.15) that (u_n) is subsequentially convergent.

If $m \equiv 1 \pmod 3$, we have that $((n\Delta)_0 V_n^{(0)}(\Delta u)) = (V_n^{(0)}(\Delta u)) = (\sigma_n^{(1)}(\omega^{(0)}(u)))$ is subsequentially convergent and then

$$V_n^{(0)}(\Delta u) = O(1). \tag{2.16}$$

Clearly, the condition (2.16) implies (2.14).

Again it follows from the conditions $\Delta V_n^{(0)}(\Delta u) = o(1)$, (2.14) and (2.15) that (u_n) is subsequentially convergent.

If $m \equiv 2 \pmod 3$, we conclude that $(n\Delta V_n^{(1)}(\Delta u)) = (\sigma_n^{(1)}(\omega^{(1)}(u)))$ is subsequentially convergent and then

$$n\Delta V_n^{(1)}(\Delta u) = O(1). \tag{2.17}$$

Clearly, the condition (2.17) implies (2.14).

From the conditions $\Delta V_n^{(0)}(\Delta u) = o(1)$, (2.14) and (2.15) it follows that (u_n) is subsequentially convergent. □

Proof of Theorem 2.2. Assume that $(V_n^{(0)}(\Delta u)) \in U(\ell_\infty)$ is a subsequential convergence condition for L. Since $(V_n^{(0)}(\Delta u)) \in U(\ell_\infty)$, by similar calculations in the proof of Theorem 2.1 we have $(n\Delta V_n^{(1)}(\Delta u)) \in \ell_\infty$, or equivalently $n\Delta V_n^{(1)}(\Delta u) = \sigma_n^{(1)}(\omega^{(1)}(u)) = O(1)$. Let $(V_n^{(m)}(\Delta u)) \in U(\ell_\infty^{(m)})$. Then by Lemma 2.5, $(\sigma_n^{(1)}(\omega^{(m+1)}(u))) \in \ell_\infty$, or equivalently

$$\sigma_n^{(1)}(\omega^{(m+1)}(u)) = n\Delta V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(m-1)}(u))) = O(1). \tag{2.18}$$

Since $L - \lim_n u_n = s$,

$$L - \lim_n \sigma_n^{(1)}(\omega^{(m-1)}(u)) = 0. \tag{2.19}$$

The condition $\Delta V_n^{(0)}(\Delta u) = o(1)$ implies that

$$\Delta((n\Delta)_m V_n^{(m)}(\Delta u)) = \Delta V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(m-1)}(u))) = o(1). \tag{2.20}$$

Taking into account (2.18), (2.19), and (2.20), we conclude that $(\sigma_n^{(1)}(\omega^{(m-1)}(u)))$ is subsequentially convergent, and then $\sigma_n^{(1)}(\omega^{(m-1)}(u)) = O(1)$, or equivalently

$$\sigma_n^{(1)}(\omega^{(m-1)}(u)) = n\Delta V_n^{(1)}(\Delta\sigma^{(1)}(\omega^{(m-3)}(u))) = O(1). \tag{2.21}$$

As in obtaining (2.19) and (2.20), we have

$$\begin{gathered} L - \lim_n \sigma_n^{(1)}(\omega^{(m-3)}(u)) = 0, \\ \Delta((n\Delta)_{m-2} V_n^{(m-2)}(\Delta u)) = \Delta V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(m-3)}(u))) = o(1). \end{gathered} \tag{2.22}$$

Taking into account (2.21) and (2.22), we conclude that from the assumption $(\sigma_n^{(1)}(\omega^{(m-3)}(u)))$ is subsequentially convergent. Continuing in this manner, if $m \equiv 0 \pmod 2$, we have $(n\Delta V_n^{(1)}(\Delta u)) = (\sigma_n^{(1)}(\omega^{(1)}(u)))$ is subsequentially convergent and then,

$$n\Delta V_n^{(1)}(\Delta u) = O(1). \tag{2.23}$$

Since $L - \lim_n u_n = s$, we have

$$L - \lim_n \sigma_n^{(1)}(\omega^{(1)}(u)) = 0. \tag{2.24}$$

It follows from the condition $\Delta V_n^{(0)}(\Delta u) = o(1)$, (2.23), and (2.24) that (u_n) is subsequentially convergent.

If $m \equiv 1 \pmod 2$, we have that $((n\Delta)_0 V_n^{(0)}(\Delta u)) = (V_n^{(0)}(\Delta u)) = (\sigma_n^{(1)}(\omega^{(0)}(u)))$ is subsequentially convergent, and then, we have

$$V_n^{(0)}(\Delta u) = O(1). \tag{2.25}$$

The condition (2.25) implies (2.23).

Taking into account $\Delta V_n^{(0)}(\Delta u) = o(1)$, (2.23), and (2.24), we have that (u_n) is subsequentially convergent. □

Proof of Theorem 2.3. Assume that $(u_n) \in U(\mathcal{M})$ is a subsequential convergence condition for L. Since $(u_n) \in U(\mathcal{M})$, by similar reasoning in the proof of Theorem 2.1, we have $(V_n^{(0)}(\Delta u)) \in \mathcal{M}$. Thus, we have $n\Delta V_n^{(1)}(\Delta u) = O(1)$. The rest of the proof is as in the proof of Theorem 2.2. □

Proof of Theorem 2.4. Assume that $(u_n) \in U(\ell_\infty)$ is a subsequential convergence condition for L. Since $(u_n) \in U(\ell_\infty)$, we have $u_n = b_n + \sum_{k=1}^{n}(b_k/k)$ for some $(b_n) \in \ell_\infty$. Thus $V_n^{(0)}(\Delta u) = O(1)$. Let $(V_n^{(m)}(\Delta u)) \in U(\ell_\infty^{(m)})$. By Lemma 2.5, $(\sigma_n^{(1)}(\omega^{(m+1)}(u))) \in \ell_\infty$, or equivalently

$$\sigma_n^{(1)}(\omega^{(m+1)}(u)) = V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(m)}(u))) = O(1). \tag{2.26}$$

$L - \lim_n u_n = s$ implies

$$L - \lim_n \sigma_n^{(1)}(\omega^{(m)}(u)) = 0 \tag{2.27}$$

and from $\Delta V_n^{(0)}(\Delta u) = o(1)$, we have

$$\Delta\big((n\Delta)_{m+1} V_n^{(m+1)}(\Delta u)\big) = \Delta V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(m)}(u))) = o(1). \tag{2.28}$$

Taking into account (2.26), (2.27), and (2.28), we conclude that $(\sigma_n^{(1)}(\omega^{(m)}(u)))$ is subsequentially convergent, and then $\sigma_n^{(1)}(\omega^{(m)}(u)) = O(1)$, or equivalently

$$\sigma_n^{(1)}(\omega^{(m)}(u)) = V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(m-1)}(u))) = O(1). \tag{2.29}$$

As in obtaining (2.27) and (2.28), we have

$$\begin{gathered} L - \lim_n \sigma_n^{(1)}(\omega^{(m-1)}(u)) = 0, \\ \Delta\big((n\Delta)_m V_n^{(m)}(\Delta u)\big) = \Delta V_n^{(0)}(\Delta\sigma^{(1)}(\omega^{(m-1)}(u))) = o(1). \end{gathered} \tag{2.30}$$

Again taking into account (2.29) and (2.30), from the assumption we obtain that $(\sigma_n^{(1)}(\omega^{(m-1)}(u)))$ is subsequentially convergent. Continuing in this manner we have that $(\sigma_n^{(1)}(\omega^{(0)}(u)))$ is subsequentially convergent, and then

$$\sigma_n^{(1)}(\omega^{(0)}(u)) = V_n^{(0)}(\Delta u) = O(1). \tag{2.31}$$

Since (u_n) is L-limitable to s, we have

$$L - \lim_n V_n^{(0)}(\Delta u) = 0. \tag{2.32}$$

From the condition $\Delta V_n^{(0)}(\Delta u) = o(1)$, (2.31), and (2.32), we conclude that (u_n) is subsequentially convergent. □

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İbrahim Çanak: Department of Mathematics, Adnan Menderes University, 09010 Aydin, Turkey
Email address: icanak@adu.edu.tr

Ümit Totur: Department of Mathematics, Adnan Menderes University, 09010 Aydin, Turkey
Email address: utotur@adu.edu.tr

Mehmet Dik: Department of Mathematics, Rockford College, 5050 E. State Street, Rockford, IL 61108, USA
Email address: mdik@rockford.edu